UNCLASSIFIED

Defense Technical Information Center Compilation Part Notice

ADP010905

TITLE: A Fractional Brownian Motion Model of Cracking

DISTRIBUTION: Approved for public release, distribution unlimited

This paper is part of the following report:

TITLE: Paradigms of Complexity. Fractals and Structures in the Sciences

To order the complete compilation report, use: ADA392358

The component part is provided here to allow users access to individually authored sections of proceedings, annals, symposia, ect. However, the component should be considered within the context of the overall compilation report and not as a stand-alone technical report.

The following component part numbers comprise the compilation report:

ADP010895 thru ADP010929

UNCLASSIFIED

A FRACTIONAL BROWNIAN MOTION MODEL OF CRACKING

P.S. ADDISON, L.T. DOUGAN, A.S. NDUMU, W. M.C. MACKENZIE

Civil Engineering Group, School of the Built Environment, Napier University, Merchiston Campus, 10 Colinton Road, Edinburgh, EH10 5DT, Scotland, UK.

An attempt is made to find the fractal cutoff of crack profiles on the tension face of concrete beams subjected to uni-axial bending. Previous work by the authors has shown that such cracking can be interpreted as a non-Fickian diffusive phenomenon resulting from a self-affine random fractal process: specifically fractional Brownian motion (fBm). In addition, a spatial description of the cracking geometry can be found from experimental data using both a (Hurst) scaling exponent and a diffusion-type coefficient. Herein the authors find that the fractal description of the crack profiles extends down to less than 0.75μm. The use of a scanning electron microscope to probe the crack profile (and surface) at smaller scales is discussed and the synthesis of crack surfaces using fBm is described briefly.

1 Introduction

An understanding of the behaviour of cracking in structural elements is of great importance in the analysis and subsequent safe design of engineering structures. As yet, however, there is no definitive theoretical framework for the propagation of cracks and resulting fracture energy. It has recently been found that the irregular geometries of both crack surfaces and crack profiles in a variety of materials may be described (and subsequently modelled) using fractal geometry. The use of fractal geometry to describe cracking phenomena is now widespread (e.g. see references[1-4]). Previous work by the authors[5,6] has shown that crack profiles on concrete beams in tension can be modelled as fBm trace functions which require a Hurst exponent and a spatial diffusion coefficient to completely describe the spatial distribution of the crack. In addition, the authors have linked the fBm description of the cracking phenomena to an effective Fokker-Planck equation which described the diffusive nature of the cracking phenomena through space. In this paper the cracks are investigated at higher resolutions in an attempt to determine whether a fractal cut-off scale exists. The value of such a Euclidean threshold is of significant importance in the determination of the energy of fracture.

2 The Diffusive Nature of fBm

Fractional Brownian Motion (fBm) is a generalisation of Brownian motion suggested by Mandelbrot[7] which has found a variety of uses in the natural sciences (see for example Addison and Ndumu[8] and the references contained therein). Fractional Brownian motion is defined as:

$$y(x) = \frac{1}{\Gamma\left(H+\frac{1}{2}\right)}\left\{\int_{-\infty}^{0}\left[(x-x')^{H-\frac{1}{2}}-(-x')^{H-\frac{1}{2}}\right]dW(x')+\int_{0}^{x}(x-x')^{H-\frac{1}{2}}\,dW(x')\right\} \tag{1}$$

where $dW(x)$ is a Gaussian random function with zero mean and unit standard deviation, H is the Hurst exponent[9], and Γ is the gamma function. When H=0.5 Eq. (1) models classical Brownian motion which produces normal, or Fickian, diffusion. From Eq. (1) it may be seen that the fBm process is correlated over all length scales, i.e., it has an infinite memory associated with it.

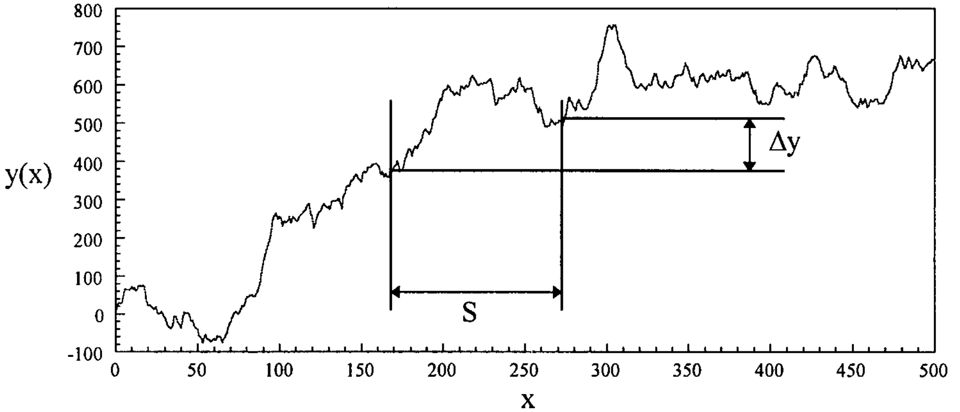

Figure 1: Diffusive scaling of an fBm ($H=0.75$ $K_f=10$)

An example of a superdiffusive fBm (i.e. one with $H > 0.5$) is shown in Fig. 1. The diffusive scaling of the fBm process shown in Fig. 1 may be defined as

$$\sigma_y = \sqrt{2K_f}\; s^H \tag{2}$$

where σ_y is the standard deviation of the y-excursions (Δy) on the trace for a window length s; K_f is a fractal diffusion coefficient. Eq. (2) is in fact the standard deviation of the probability density function

$$P(y,x) = \frac{1}{\sqrt{4\pi K_f x^{2H}}} \exp\left(-\frac{y^2}{4K_f x^{2H}}\right) \tag{3}$$

which is a non-Fickian scaling of a Gaussian probability density function through space. (If $H = ½$ then Eq. (3) reduces to the solution of a Fickian based diffusion from a point source.) Furthermore, it has been shown by Wang and Lung[10] that Eq. (3) is the solution to the effective Fokker-Planck equation:

$$\frac{\partial P(y,x)}{\partial x} = 2H\; K_f\; x^{2H-1}\, \frac{\partial^2 P(y,x)}{\partial y^2} \tag{4}$$

which describes the probability of occurrence of $y(x)$ at spatial location x. Eq. (4) is in fact a generalisation of the classical Fickian diffusion equation with a spatial diffusion coefficient. The equation reduces to the classical equation for H=0.5.

It can be seen from the above that, over a large number of realisations, fBm approximates a non-Fickian diffusive process described by Eq. (4). The authors have previously shown that persistent fBm ($H > 0.5$) is a suitable model for cracking on the tension face of a concrete beam in bending[5]. In addition, both H and K_f are required for a complete geometric description of the cracking phenomena. It was shown by the authors how these parameters can be found from experiment. The mean values of K_f and H were found for a series of flexure cracks in concrete beams to be 0.084 and 0.77, respectively (i.e. superdiffusive surfaces with fractal dimensions between 1 and 1.5). This gives the standard deviation of the cracking across the beam as $\sigma_y = \sqrt{2 \times 0.084}\, s^{0.77}$ where both s and σ_y are expressed in millimetres. Thus, for the 40mm wide specimens used in the study the expected standard deviation of the crack displacement across the beam is 7.02mm. These experimentally derived parameters can be used to synthesise crack patterns using fBms. An example of this is shown in Fig. 2 using the fBm generation method described by Addison *et al.*,[11]. In the figure a crack with measured values of H and K_f of 0.75 and 0.133 respectively is shown together with a synthesised crack with the same parameter values. The similarity between the two traces is evident from a visual inspection of the plot.

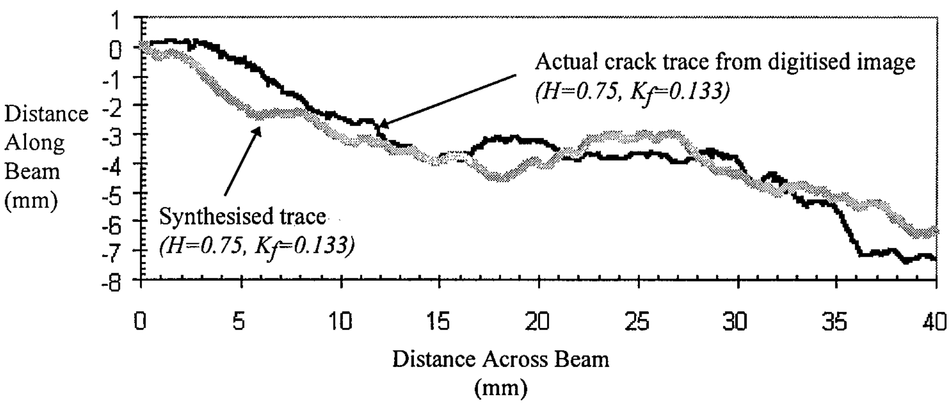

Figure 2: Comparison of synthesised and experimental cracks

3 Crack Profile Analysis

Natural fractals tend to exhibit fractal characteristics over a limited range of scales[12]. Below a cut-off level, the natural fractal object tends to revert to a Euclidean form. The authors have recently pursued the search for the cut-off length scale in the crack patterns as it has implications for the measurement of the true areas of crack surfaces and hence energy dissipation across the surface. Fig. 3 shows one of the cracks studied by the authors at a magnification level of 6×. Seven boxes are placed on the crack profile indicating regions where a closer inspection was taken of the crack profile at the higher magnification of 40×. In addition, two smaller boxes indicate locations where the crack was studied at 50× and 100× magnification, respectively. Fig. 4 contains a log-log plot of σ_y against s. From such a plot it is possible to calculate both H and K_f[6]. A line of slope H=1 is also given in the plot, corresponding to a dimension of unity, i.e. a smooth Euclidean curve. It would be expected that the plotted curves tend to this slope at the Euclidean cut-off.

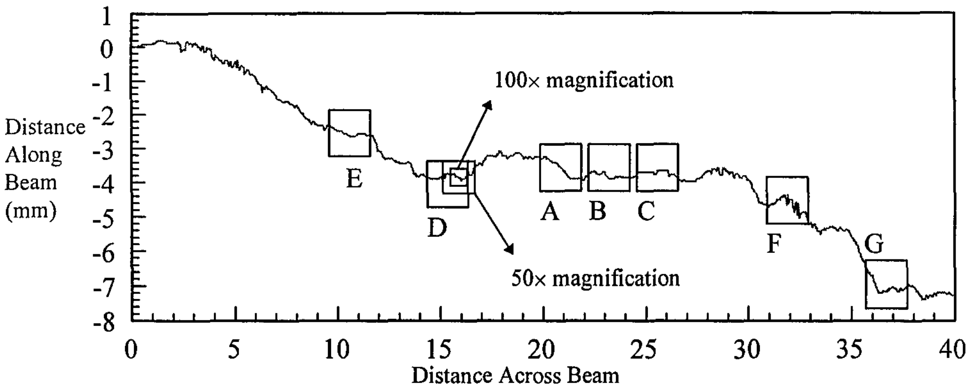

Figure 3: Crack profile showing analysed regions

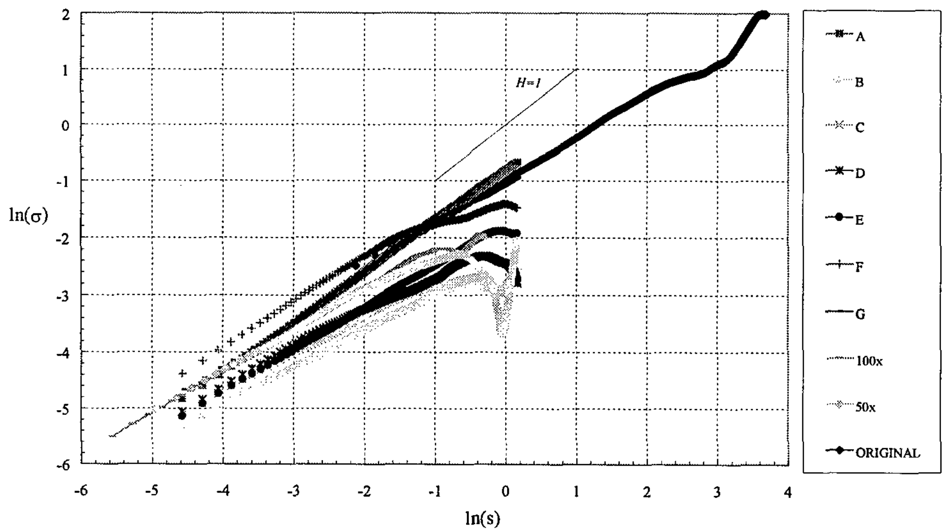

Figure 4: Logarithmic plots of σ_s versus s for the crack shown in figure 3. The graph shows (extending out, bottom left) two sections representing 50× and 100× magnifications.

Table 1 contains the H and K_f values for the various regions of the crack in Fig. 3 measured from Fig. 4. It can be seen from the table that measured H values for the whole crack and the average values from the seven boxes A to G are in good agreement. However, the K_f values between the two are significantly different. In fact the K_f values vary over a large range (0.004-0.118) across the selected boxes. The reason for this variability is as yet unclear. Zooming in at 50× and 100× again produces persistent values of the Hurst exponent. It can be seen from the plot that the fractal description of the curves extends down to the lower limits of the 100× magnification: this relates to a resolution of 7.5×10^{-4}mm of crack per pixel.

Table1: The Measured H and K_f Values for the Crack in Figure 3.

	H	K_f
Whole Crack (6×)	**0.74**	**0.069**
A (40×)	0.87	0.093
B (40×)	0.64	0.004
C (40×)	0.68	0.008
D (40×)	0.61	0.018
E (40×)	0.72	0.015
F (40×)	0.82	0.118
G (40×)	0.83	0.091
Average (A-G)	**0.74**	**0.050**
50×	**0.71**	**0.025**
100×	**0.74**	**0.030**

4 Concluding Remarks

From Fig. 4, the lower limit to the fractal behaviour of the cracking patterns (if it exists!) appears to be below 0.75μm. This is significantly less than the value between 10 and 20 μm (which is the approximate size of calcium silicate hydrate) suggested by Souma and Barton[4]. The authors have recently initiated research to search for a fractal cut-off scale at higher resolutions at the crack edge using a scanning electron microscope (e.g. Fig. 5). This work has so far proved inconclusive due to the difficulty in finding reasonable vertical sections through the crack edge at these higher scales. It is hoped that an improvement in the experimental techniques will lead to a better understanding of the crack geometry at these smaller scales.

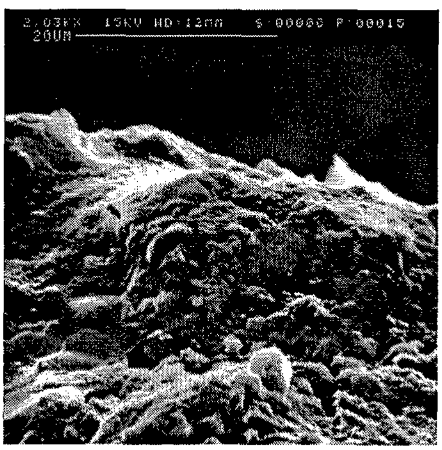

Figure 5: An electron microscope image of the crack profile at 2000× magnification. Note the difficulty in defining the edge of the crack.

The determination of the true fractal cut-off scale has implications for the synthesis of prefractal fBm crack profiles and surfaces and hence the calculation of crack energy across the crack surface. If the crack surface is in fact fBm, then the crack profile may be treated as the result of a vertical cut through the surface[13]. If this proves to be the case, then it is known that the fractal dimension of the surface $D_{surface}$ is equal to $D_{profile}$+1. It is relatively simple to generate such a surface using a variety of methods. It should be possible therefore to synthesise the crack surface using the K_f and H values found from experiment. Fig. 6 shows an fBm surface generated using the turning bands method[14]. The authors intend to pursue the measurement and synthesis of crack profiles and surfaces in order to define the fracture energy of cracking in terms of a fractal geometric framework based on fBms.

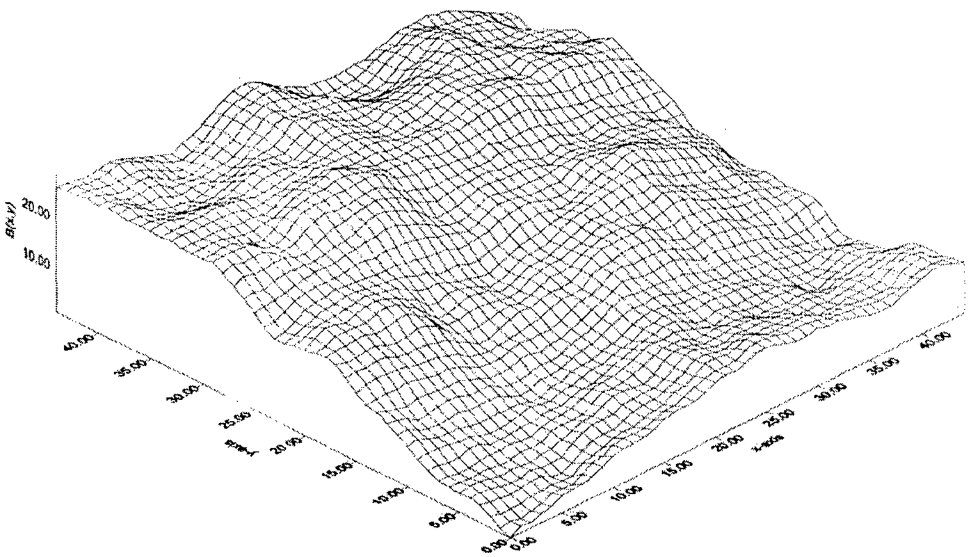

Figure 6: An fBm surface generated using the turning bands method (*H=0.8*)

References

1. Carpinteri A. (1994a). 'Scaling laws and renormalization groups for strength and toughness of disordered materials.' *Int. J. Solids Struct.*, 31(3), 291-302.
2. Carpinteri A. (1994b). 'Fractal nature of material microstructure and size effects on apparent mechanical properties.' *Mechanics of Materials*, 18, 89-101.
3. Borodich F.M. (1997). 'Some fractal models of fracture.' *J. Mech. Phys. Solids*, 45(2), 239-259.
4. Saouma V.E., Barton C.C. and Gamaleldin N.A. (1990). 'Fractal characterisation of fracture surfaces in concrete.' *Engng. Fract. Mech.*, 35, 47-53.

5. Addison P.S., MacKenzie W.M.C., Ndumu A.S., Dougan L. and Hunter R. (1999), 'Fractal Cracking of Concrete: Parameterisation of Spatial Diffusion', *ASCE J. Engng. Mech.*, 125(6), 622-629.
6. Addison P.S., Dougan L.T., Ndumu A.S and Mackenzie W.M.C. (1999). 'A Complete Geometric Description of Cracked Concrete', 13th ASCE Engineering Mechanics Division Conference, Baltimore, MD, USA, June 13-16.
7. Mandelbrot B.B. & Wallis J.R. (1969). 'Computer experiments with fractional Gaussian noises. Part 3, Mathematical Appendix.' *Water Resour. Res.*, 5, 260-267.
8. Addison P.S. and Ndumu A.S. (1999), 'Engineering Applications of Fractional Brownian Motion: Self-Similar and Self-Affine Random Processes', *Fractals,* 7(2), 151-157.
9. Hurst H.E. (1951). 'Long term storage capacity of reservoirs', *Trans. Am. Soc. Civil Eng.*, 116, 770-808.
10. Wang K.G., and Lung, C.W. (1990). 'Long-time correlation effects and fractal Brownian motion.' *Phys. Letts. A*, 151(3,4), 119-121.
11. Addison P.S., Qu B, Ndumu A.S. and Pyrah I.C. (1998). 'A particle tracking model for non-Fickian subsurface diffusion.' *Math. Geology*, 30(6), 695-716.
12. Bunde A & Havlin S., eds. (1994). Fractals in science. Springer-Verlag, Berlin.
13. Addison P.S. (1997). Fractals and chaos: An illustrated course. Institute of Physics Publishing, Bristol.
14. Mantoglou, A., and Wilson, J. L. (1982). The turning bands method for simulation of random fields using line generation by a spectral method. *Water Resour. Res.*, 18(5), 1379-1394.